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Structural Similitude and Scaling Laws for Sandwich Beams

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Problems associated primarily with structural similitude as applied to sandwich beams and columns are investigated. Similitude theory is employed to develop the necessary similarity conditions or scaling laws. Scaling laws provide the relationship between a full-scale structure (prototype) and its scaled down model. These relationships can be used to design the small-scale model and to extrapolate the experimental data of the model to predict the behavior of the large prototype. Note that scaled down models are inexpensive and easily tested. The developed methodology is demonstrated through analysis of the bending of sandwich beams subjected to transverse loads, and it is totally analytical. A small model is designed and theoretically analyzed. The theoretically obtained response predictions are treated as if they were experimentally obtained (from model testing). Then, scaling laws are used with the model data to predict the behavior of the large prototype. Finally, these predictions are compared to the theoretically obtained response of the prototype to test the validity of the method. Predicted and theoretical response is very close, and therefore, the demonstration is successful. Both complete (satisfaction of all similarity conditions) and partial similarity are discussed.

Nomenclature

bw_k	=	width of panel
c_k	=	thickness of the core
dt_k, db_k	=	thickness of the upper and the lower skin, respectively
EAt_k, EAb_k	=	longitudinal rigidities of the upper and the lower face, respectively
Ec_k, Gc_k	=	vertical modulus of elasticity of the core and its shear modulus, respectively
EI_k	=	flexural rigidity of beam
EIt_k, EIb_k	=	flexural rigidities of upper and lower faces, respectively
k	=	p or m, where p refers to the prototype and m to the model
L	=	length of the beam
N_{tot}, M_{tot}, Q_{tot}	=	global longitudinal force, bending moment, and shear force stress resultant
N_{xxt}, M_{xxt}, Q_{xxt}; N_{xxb}, M_{xxb}, Q_{xxb}	=	longitudinal force, bending moment, and shear force stress resultants in upper and lower faces, respectively
nt_k, qt_k, mt_k; nb_k, qb_k, mb_k	=	external longitudinal and vertical distributed loads and distributed bending moments exerted at the upper and the lower faces, respectively
Q_{xxc}	=	vertical shear stress resultant in core
q_k	=	external distributed load on beam
$Uot_k(x_k)$, $Uob_k(x_k)$	=	in-plane deformations of the upper and the lower faces, respectively
$Wt_k(x_k)$, $Wb_k(x_k)$	=	vertical deformations of the upper and the lower faces, respectively
$wc_k(x_k, z_k)$, $uc_k(x_k, z_k)$	=	vertical and longitudinal displacements fields in core
$w_k(x_k)$	=	vertical displacement of beam
X_j^K	=	j parameter or response variable
x_k	=	longitudinal coordinate of beam/panel
z_k	=	vertical coordinate of core measured from upper face–core interface
$\bar{z}_t$, $\bar{z}_b$	=	distances of the centroid of the upper and the lower face sheets from the centroid of the sandwich panel
λ_j	=	scale factor of the j parameter
σ_{zzt_k}, σ_{zzb_k}	=	vertical normal stresses at upper and lower face–core interfaces, respectively
$\sigma_{zz_k}(x_k, z_k)$	=	vertical normal stress field of core
$\tau_k(x_k)$	=	vertical shear stress in the core

Introduction

THERE is renewed interest in the use of sandwich construction, especially for future large transport aircraft. Sandwich structures can be attractive for such airplanes due to their high specific strength and stiffness, as well as due to damage tolerance. The correct and effective use of sandwich construction with laminated composite facings requires sophisticated, complex analyses to achieve good understanding of the system response characteristics to external causes. Because of the complexity of the systems and the lack of complete design-based information, any new design must be extensively evaluated experimentally until it achieves the necessary reliability, performance, and safety. However, the experimental evaluation of these structures is extremely useful if a similar small-scaled model can replace a full-scale structure, which is much easier to work with. Furthermore, a dramatic reduction in cost and time can be achieved if available experimental data of a specific structure can be used to predict the behavior of a group of similar systems.

To understand the applicability of these models in designing laminated composite and sandwich structures, an analytical investigation was undertaken to assess the feasibility of their use. Employment of similitude theory to establish similarity among structural systems can save considerable expense and time provided the proper scaling laws are found and validated.

Some studies concerning the use of scaled down shell structures have been conducted in the past. Ezra[1] presented a study, based on dimensional analysis, for buckling behavior of scaled down shell structures subjected to impulsive loads. Morgen[2] presented a similar investigation for an orthotropic cylinder subjected to different static loads. Because of the large number of design parameters, the

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identification of the conventional method of dimensional analysis is tedious. Similitude theory based on governing equations of the structural system is more direct and simpler in execution. Additional studies have been reported in the past 15–20 years on structural similitude and modeling. See Simitses et al.[3] for a comprehensive review.

Mathematical Formulation

The mathematical formulations outline the procedure that yields the similitude relations. The approach is based on matching of the coefficients of the differential governing equations of the prototype equations with those of the models equations. The approach has been implemented through symbolic algebraic interpreter software, Maple V R6,[4] and the results are presented.

The similitude approach assumes that there is a relation between all of the parameters and response terms as follows:

$$\lambda_j = X_j^p / X_j^m \tag{1}$$

where X_j^K is the j parameter or response variable of the prototype p or the model m.

The approach of matching terms of the governing differential equations is demonstrated first for bending of beams for simplicity and to show the approach. Later a sandwich panel similarity analysis is presented. In all cases, the formulation and the results are determined analytically, and in the sandwich case, it is followed by some numerical examples.

Similitude of a Beam in Bending Only

The governing differential equation of the isotropic prototype using small deformations is

$$EI_p\left[\frac{\partial^4}{\partial x_p^4} w_p(x_p)\right] - q_p = 0 \tag{2}$$

where EI_p is the flexural rigidity, q_p is the external distributed load, $w_p(x_p)$ is the vertical displacement, and x_p is coordinate of the beam between 0 and L, which is the length of the beam.

The relation between the prototype parameters and those of the model, or in other words, the description of the prototype parameters in terms of the scale factors and the model parameters, is

$$EI_p = \lambda_{EI} EI_m,\ w_p(x_p) = \lambda_w w_m(x_m), \qquad q_p = \lambda_q q_m$$
$$x_p = \lambda_x x_m \tag{3}$$

Substitution of these parameters in Eq. (3) leads to the governing equation of the prototype in terms of the model parameters, denoted by subscript m. After dividing by λ_q it is

$$\frac{\lambda_{EI} EI_m \lambda_w \left[\left(\partial^4/\partial x_m^4\right) w_m(x_m)\right]}{\lambda_q \lambda_x^4} - q_m = 0 \tag{4}$$

The governing equation of the model reads

$$EI_m\left[\frac{\partial^4}{\partial x^4} w_m(x_m)\right] - q_m = 0 \tag{5}$$

For the response of the model to match that of the prototype, provided that the boundary conditions are identical, the coefficients of the two equations. Eqs. (4) and (5), must be identical. Hence, by matching the coefficients of the two equations, the two similarity conditions read

$$EI_m = \frac{\lambda_{EI} EI_m \lambda_w}{\lambda_q \lambda_x^4}, \qquad q_m = q_m \qquad \text{or} \qquad 1 = \frac{\lambda_{EI}\lambda_w}{\lambda_q \lambda_x^4} \tag{6}$$

or

$$\lambda_w = \frac{\lambda_q \lambda_x^4}{\lambda_{EI}} \tag{7}$$

Hence, the deformation scale factor is proportional to the load and to the length scale factor by the power of four and inversely proportional to the bending rigidity scale factor.

Next, we discuss the similarity approach to sandwich panels with a soft core and isotropic faces. The formulation is based on the high-order modeling of such a structure; for more details on the high-order approach see Frostig et al.[5]

Similarity Analysis of Sandwich Panel with Transversely Flexible Core in Bending: High-Order Model

Next, we discuss the similarity approach to sandwich panels with a soft core and isotropic face. The formulation is based on the high-order modeling of such structure. For more details on the high-order approach, see Frostig et al.,[5] and for applications and verifications, see Refs. 6–16. The basic assumptions of the high-order formulation consist of treating the face sheets as Bernoulli–Euler beams with negligible shear and vertical normal through the thickness deformation and with small deformations. The core is considered a two-dimensional elastic medium with shear and vertical normal stresses, whereas the longitudinal stresses in the core are neglected. The longitudinal stresses' deformation fields are results of analytical solutions. The core–face interfaces provide full bond, and the loads are applied to the faces only. For more details see Frostig et al.[5]

The governing equations of a sandwich panel with isotropic faces and a transversely flexible core are[5]

$$EAt_p\left[\frac{\partial^2}{\partial x_p^2} Uot_p(x_p)\right] + bw_p\tau_p(x_p) + nt_p(x_p) = 0 \tag{8}$$

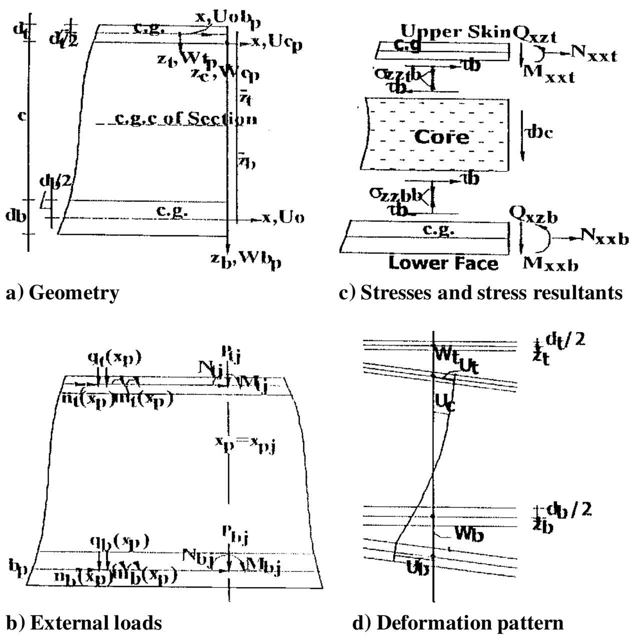
$$EAb_p\left[\frac{\partial^2}{\partial x_p^2} Uob_p(x_p)\right] - bw_p\tau_p(x_p) + nb_p(x_p) = 0 \tag{9}$$

$$\begin{aligned}&-EIt_p\left[\frac{\partial^4}{\partial x_p^4} Wt_p(x_p)\right]\\ &\quad + bw_p\left\{\frac{[Wb_p(x_p) - Wt_p(x_p)]Ec_p}{c_p} + \frac{1}{2}c_p\left[\frac{\partial}{\partial x_p}\tau_p(x_p)\right]\right\}\\ &\quad + \frac{1}{2}bw_p\,\mathrm{d}t_p\left[\frac{\partial}{\partial x_p}\tau_p(x_p)\right] + qt_p - \left[\frac{\partial}{\partial x_p} mt_p(x_p)\right] = 0\end{aligned} \tag{10}$$

$$\begin{aligned}&-EIb_p\left[\frac{\partial^4}{\partial x_p^4} Wb_p(x_p)\right]\\ &\quad - bw_p\left\{\frac{[Wb_p(x_p) - Wt_p(x_p)]Ec_p}{c_p} - \frac{1}{2}c_p\left[\frac{\partial}{\partial x_p}\tau_p(x_p)\right]\right\}\\ &\quad + \frac{1}{2}bw_p\,\mathrm{d}b_p\left[\frac{\partial}{\partial x_p}\tau_p(x_p)\right] + qb_p - \left[\frac{\partial}{\partial x_p} mb_p(x_p)\right] = 0\end{aligned} \tag{11}$$

$$\begin{aligned}&-\frac{c_p\tau_p(x_p)}{Gc_p} - Uot_p(x_p) - \left(-\frac{1}{2}c_p - \frac{1}{2}\mathrm{d}t_p\right)\left[\frac{\partial}{\partial x_p} Wt_p(x_p)\right]\\ &\quad - \left(-\frac{1}{2}c_p - \frac{1}{2}\mathrm{d}b_p\right)\left[\frac{\partial}{\partial x_p} Wb_p(x_p)\right] + Uob_p(x_p)\\ &\quad + \frac{\frac{1}{12}c_p^3\left[\left(\partial^2/\partial x_p^2\right)\tau_p(x_p)\right]}{Ec_p} = 0\end{aligned} \tag{12}$$

where EAt_p and EAb_p are the longitudinal rigidities of the upper and the lower face, respectively; EIt_p and EIb_p are the flexural rigidities of the various skins; bw_p and c_p are the width of the panel and thickness of the core; $\mathrm{d}t_p$ and $\mathrm{d}b_p$ are the thickness of the upper and the lower skin, respectively; Ec_p and Gc_p are the vertical modulus of elasticity of the core and its shear modulus; and x_p is the longitudinal coordinate of the panel. The external loads nk_p, qk_p, and mk_p, $k = t$ and b, are the longitudinal and vertical distributed loads and distributed bending moments exerted at the upper and the lower faces, respectively. The response results, $Uot_p(x_p)$ and $Uob_p(x_p)$, are the longitudinal deformations of the centroid of the upper and



a) Geometry **c) Stresses and stress resultants**

b) External loads **d) Deformation pattern**

Fig. 1 Sandwich panel.

the lower faces, respectively. $Wt_p(x_p)$ and $Wb_p(x_p)$ are the vertical deformations of the various faces, and $\tau_p(x_p)$ is the shear stress in the vertical direction of the core. For sign conventions, loads, internal stress resultants and stresses at face–core interfaces, and deformation patterns, see Fig. 1.

The stress and the deformation fields in the core and at the face–core interfaces are as follows. (For details see Frotig et al.[5])

The vertical normal stresses at the upper and the lower face–core interfaces (Fig. 1c) are

$$\sigma_{zzt_p} = \frac{[Wb_p(x_p) - Wt_p(x_p)]Ec_p}{c_p} + \frac{1}{2}c_p\left[\frac{\partial}{\partial x_p}\tau_p(x_p)\right] \quad (13)$$

$$\sigma_{zzb_p} = \frac{[Wb_p(x_p) - Wt_p(x_p)]Ec_p}{c_p} - \frac{1}{2}c_p\left[\frac{\partial}{\partial x_p}\tau_p(x_p)\right] \quad (14)$$

The stress fields in the core equal

$$\tau_p(x_p, z_p) = \tau_p(x_p) \quad (15)$$

$$\sigma_{zz_p}(x_p, z_p) = \left(\frac{1}{2}c_p - z_p\right)\left[\frac{\partial}{\partial x_p}\tau_p(x_p)\right] + \frac{[Wb_p(x_p) - Wt_p(x_p)]Ec_p}{c_p} \quad (16)$$

The deformation fields through the depth of core are

$$wc_p(x_p, z_p) = \frac{\left[\frac{1}{2}c_p z_p - \frac{1}{2}z_p^2\right][(\partial/\partial x_p)\tau_p(x_p)]}{Ec_p} + \left(1 - \frac{z_p}{c_p}\right)Wt_p(x_p) + \frac{z_p Wb_p(x_p)}{c_p} \quad (17)$$

$$uc_p(x_p, z_p) = \frac{z_p\tau_p(x_p)}{Gc_p} + \frac{\left(-\frac{1}{4}c_p z_p^2 + \frac{1}{6}z_p^3\right)\left[\left(\partial^2/\partial x_p^2\right)\tau_p(x_p)\right]}{Ec_p} + Uot_p(x_p) + \left(-z_p + \frac{\frac{1}{2}z_p^2}{c_p} - \frac{1}{2}\,\mathrm{d}t_p\right)\left[\frac{\partial}{\partial x_p}Wt_p(x_p)\right] - \frac{1}{2}\frac{z_p^2[(\partial/\partial x_p)Wb_p(x_p)]}{c_p} \quad (18)$$

The relations between the properties, rigidities, and loads of the prototype and those of the model in terms of the λ are

$$bw_p = \lambda_b bw_m, \qquad dt_p = \lambda_{dt} dt_m, \qquad db_p = \lambda_{db} db_m$$
$$c_p = \lambda_c c_m, \qquad EAt_p = \lambda_{EAt} EAt_m, \qquad EAb_p = \lambda_{EAb} EAb_m$$
$$EIt_p = \lambda_{EIt} Eit_m, \qquad EIb_p = \lambda_{EIb} EIb_m, \qquad Ec_p = \lambda_{Ec} Ec_m$$
$$Gc_p = \lambda_{Gc} Gc_m, \qquad nt_p = \lambda_{nt} nt_m, \qquad nb_p = \lambda_{nb} nb_m$$
$$qt_p = \lambda_{qt} qt_m, \qquad qb_p = \lambda_{qb} qb_m, \qquad mt_p(x_p) = \lambda_{mt} mt_m(x_m)$$
$$mb_p(x_p) = \lambda_{mb} mb_m(x_m) \quad (19)$$

The relations between the longitudinal and the vertical coordinates of the two structures are

$$L_p = \lambda_x L_m, \qquad z_p = \lambda_c z_m \quad (20)$$

where L_p and L_m are the span of the prototype and of the model, respectively.

The relations between the response variables are

$$Uot_p(x_p) = \lambda_{uot} Uot_m(x_m), \qquad Uob_p(x_p) = \lambda_{uob} Uob_m(x_m)$$
$$Wt_p(x_p) = \lambda_{wt} Wt_m(x_m), \qquad Wb_p(x_p) = \lambda_{wb} Wb_m(x_m)$$
$$\tau_p(x_p) = \lambda_\tau \tau_m(x_m) \quad (21)$$

By using the procedure shown earlier, the similarity equations are derived as follows. Substitution of Eqs. (19–21) into the prototype equations (8–12) yields the governing equations of the prototype structures in terms of the model terms. Matching the coefficients of the prototype equations described by the model properties with the model equations (8–12) yields the following similarity conditions:

$$-\frac{\lambda_{mt}}{\lambda_{qt}\lambda_x} + 1 = 0, \qquad 1 - \lambda_{uot} = 0, \qquad \lambda_{uob} - 1 = 0$$
$$-\frac{1}{2} + \frac{\frac{1}{2}\lambda_b\lambda_c\lambda_\tau}{\lambda_{qt}\lambda_x} = 0, \qquad \frac{1}{2}\frac{\lambda_b\lambda_{dt}\lambda_\tau}{\lambda_{qt}\lambda_x} - \frac{1}{2} = 0$$
$$\frac{\lambda_b\lambda_{Ec}\lambda_{wb}}{\lambda_{qt}\lambda_c} - 1 = 0, \qquad -\frac{\lambda_b\lambda_{Ec}\lambda_{wt}}{\lambda_{qt}\lambda_c} + 1 = 0$$
$$-\frac{\lambda_{EIt}\lambda_{wt}}{\lambda_{qt}\lambda_{xx}^4} + 1 = 0, \qquad \frac{1}{2}\frac{\lambda_{wt}\lambda_c}{\lambda_x} - \frac{1}{2} = 0$$
$$\frac{1}{2}\frac{\lambda_{wt}\lambda_{dt}}{\lambda_x} - \frac{1}{2} = 0, \qquad -\frac{\lambda_c\lambda_\tau}{\lambda_{Gc}} + 1 = 0$$
$$\frac{1}{12}\frac{\lambda_c^3\lambda_\tau}{\lambda_x^2\lambda_{Ec}} - \frac{1}{12} = 0, \qquad \frac{1}{2}\frac{\lambda_{wb}\lambda_c}{\lambda_x} - \frac{1}{2} = 0$$
$$\frac{1}{2}\frac{\lambda_{wb}\lambda_{db}}{\lambda_x} - \frac{1}{2} = 0, \qquad -\frac{\lambda_{EIb}\lambda_{wb}}{\lambda_{qb}\lambda_x^4} + 1 = 0$$
$$\frac{\lambda_{EAt}\lambda_{uot}}{\lambda_{nt}\lambda_x^2} - 1 = 0, \qquad \frac{\lambda_b\lambda_\tau}{\lambda_{nt}} - 1 = 0, \qquad \frac{\lambda_b\lambda_{Ec}\lambda_{wt}}{\lambda_{qb}\lambda_c} - 1 = 0$$
$$-\frac{\lambda_b\lambda_{Ec}\lambda_{wb}}{\lambda_{qb}\lambda_c} + 1 = 0, \qquad \frac{\lambda_{EAb}\lambda_{uob}}{\lambda_{nb}\lambda_x^2} - 1 = 0$$
$$-\frac{\lambda_b\lambda_\tau}{\lambda_{nb}} + 1 = 0, \qquad -\frac{1}{2} + \frac{\frac{1}{2}\lambda_b\lambda_c\lambda_\tau}{\lambda_{qb}\lambda_x} = 0$$
$$\frac{1}{2}\frac{\lambda_b\lambda_{db}\lambda_\tau}{\lambda_{qb}\lambda_x} - \frac{1}{2} = 0, \qquad -\frac{\lambda_{mb}}{\lambda_{qb}\lambda_x} + 1 = 0 \quad (22)$$

The number of system parameters is 22, whereas the number of independent similarity conditions is 18. [Note that some of the 24 relations in Eqs. (22) are independent.] This means that we can freely choose our scale factor in designing the model for a given prototype.

For this particular application we choose

$$\lambda_{qt}, \qquad \lambda_{Ec}, \qquad \lambda_c, \qquad \lambda_x \tag{23}$$

Then we solve the similarity conditions by employing Maple and obtaining the remaining 18 scale factors in terms of the four in Eq. (23). These are

$$\lambda_c = \lambda_c, \qquad \lambda_{Ec} = \lambda_{Ec}, \qquad \lambda_{wb} = \frac{\lambda_{xx}}{\lambda_c}, \qquad \lambda_{db} = \lambda_c$$
$$\lambda_{wt} = \frac{\lambda_x}{\lambda_c}, \qquad \lambda_{uob} = 1, \qquad \lambda_{mt} = \lambda_{qt}\lambda_x, \qquad \lambda_{dt} = \lambda_c$$
$$\lambda_{uot} = 1, \qquad \lambda_{EIt} = \lambda_{qt}\lambda_x^3\lambda_c, \qquad \lambda_\tau = \frac{\lambda_x^2\lambda_{Ec}}{\lambda_c^3}$$
$$\lambda_b = \frac{\lambda_{qt}\lambda_c^2}{\lambda_{Ec}\lambda_{xx}}, \qquad \lambda_{qb} = \lambda_{qt}, \qquad \lambda_{EIb} = \lambda_{qt}\lambda_x^3\lambda_c$$
$$\lambda_{EAb} = \frac{\lambda_{qt}\lambda_x^3}{\lambda_c}, \qquad \lambda_{EAt} = \frac{\lambda_{qt}\lambda_x^3}{\lambda_c}, \qquad \lambda_{nb} = \frac{\lambda_{qt}\lambda_x}{\lambda_c}$$
$$\lambda_{nt} = \frac{\lambda_{qt}\lambda_x}{\lambda_c}, \qquad \lambda_{Gc} = \frac{\lambda_x^2\lambda_{Ec}}{\lambda_c^2}, \qquad \lambda_{mb} = \lambda_{qt}\lambda_x \tag{24}$$

Notice that the longitudinal displacement terms λ_{uot} and $\lambda_{uob} = 1$ and that the vertical displacement terms λ_{wt} and λ_{wb} depend on λ_x and λ_c. The thickness terms λ_{dt} and λ_{db} are related to λ_c only. Hence, in the case where the thickness of the model and its length have the same ratio with repect to the prototype dimensions, then $\lambda_x = \lambda_c$, or in other words, the model is actually a scaled down description of the prototype. Hence, the following similarity conditions exist:

$$\lambda_c = \lambda_c, \qquad \lambda_{Ec} = \lambda_{Ec}, \qquad \lambda_{wb} = \frac{\lambda_{xx}}{\lambda_c}, \qquad \lambda_{db} = \lambda_c$$
$$\lambda_{wt} = 1, \qquad \lambda_{uob} = 1, \qquad \lambda_{mt} = \lambda_{qt}\lambda_c, \qquad \lambda_{dt} = \lambda_c$$
$$\lambda_{uot} = 1, \qquad \lambda_{EIt} = \lambda_{qt}\lambda_c^4, \qquad \lambda_\tau = \frac{\lambda_{Ec}}{\lambda_c}$$
$$\lambda_b = \frac{\lambda_{qt}\lambda_c^2}{\lambda_{Ec}\lambda_{xx}}, \qquad \lambda_{qb} = \lambda_{qt}, \qquad \lambda_{EIb} = \lambda_{qt}\lambda_c^4$$
$$\lambda_{EAb} = \lambda_{qt}\lambda_c^2, \qquad \lambda_{EAt} = \lambda_{qt}\lambda_c^2, \qquad \lambda_{nb} = \lambda_{qt}$$
$$\lambda_{nt} = \lambda_{qt}, \qquad \lambda_{Gc} = \lambda_{Ec}, \qquad \lambda_{mb} = \lambda_{qt}\lambda_c \tag{25}$$

where the deformation simlarity terms equal 1 and the shear modulus term λ_{Gc} in the core equals that of the vertical modulus of the core, λ_{Ec}, which is an indpendent variable. Thus, in case of full geometrical similarity, the core moduli, vertical, and shear have the same ratio, and it means that the core in the model may be an isotropic one.

The scale factors for the core have been derived by substituting the similarity conditions Eqs. (25) in the stress and deformation fields of the core of the prototype [see Eqs. (13), (14), and (16–18)]. The core scale factors are

$$\lambda_{\sigma_{zzc}} = \lambda_{Ec}/\lambda_c, \qquad \lambda_{wc} = 1, \qquad \lambda_{uc} = 1 \tag{26}$$

where $\lambda_{\sigma_{zzc}}$ is the vertical normal stress ratio in the core and λ_{wc} and λ_{uc} are the similitude ratio of the vertical deformation and the longitudinal one, respectively, in the core.

The similitude factors for the stress resultants following the procedure described earlier are

$$\lambda_{Nxxt} = \lambda_{qt}\lambda_x^2/\lambda_c, \qquad \lambda_{Nxxb} = \lambda_{qt}\lambda_x^2/\lambda_c, \qquad \lambda_{Mxxt} = \lambda_{qt}\lambda_x^2$$
$$\lambda_{Mxxb} = \lambda_{qt}\lambda_x^2, \qquad \lambda_{Qxxt} = \lambda_{qt}\lambda_x$$
$$\lambda_{Qxxb} = \lambda_{qt}\lambda_x, \qquad \lambda_{Qxxc} = \lambda_{qt}\lambda_x \tag{27}$$

where $Q_{xzc} = \tau bc$ is the shear stress resultant in the core. All other stress resultants appear in Fig. 1.

The overall stress resultants in a section of the sandwich panel are

$$\lambda_{N\text{tot}} = \lambda_{qt}\lambda_x^2/\lambda_c, \qquad \lambda_{M\text{tot}} = \lambda_{qt}\lambda_x^2, \qquad \lambda_{Q\text{tot}} = \lambda_{qt}\lambda_x \tag{28}$$

where

$$N_{\text{tot}} = N_{xxt} + N_{xxb}, \qquad M_{\text{tot}} = M_{xxt} + M_{xxb} - N_{xxt}\bar{z}_t + N_{xxb}\bar{z}_b$$
$$Q_{\text{tot}} = Q_{xzt} + Q_{xzb} + Q_{xzc} \tag{29}$$

In addition, $\bar{z}_t$ and $\bar{z}_b$ are the distances of the centroid of the upper and the lower face sheets from the centroid of the sandwich panel (Fig. 1).

Next, a numerical study of some typical cases is conducted with complete and partial similarity for demonstration purposes.

Numerical Example

A prototype sandwich panel with a soft core supported at its lower face sheet at three points and loaded by a distributed symmetrical load is investigated. Geometry, mechanical properties, loading scheme, and boundary conditions appear in Fig. 2, where the face sheets rigidity of the prototype are $EA_k = E_s bd_k$ and $EI_k = E_s bd_k^3/12$ with $k = t, b$ for the upper and lower face sheets, respectively. Note that in the mathematical formulation these rigidities are independent and each one has its own scale factor. A few cases are described next with complete and partial similarity.

Complete Similarity (Case 1)

The first model designed for the prototype on Fig. 2 consists of the following scale factors with full geometry reduction of five, that is, the model is a scaled down description of the prototype by the order of five, the load is reduced by a factor of 10, and the modulus of the core is reduced by the order of 10; hence,

$$\lambda_c = 5, \qquad \lambda_{qt} = 10, \qquad \lambda_{Ec} = 10, \qquad \lambda_x = 5 \tag{30}$$

These are the freely chosen scale factors. The remaining scale factors are obtained from Eqs. (25):

$$\lambda_{wt} = 1, \quad \lambda_\tau = 2, \quad \lambda_{wb} = 1, \quad \lambda_{uob} = 1, \quad \lambda_{uot} = 1$$
$$\lambda_{dt} = 5, \quad \lambda_b = 5, \quad \lambda_{Gc} = 10, \quad \lambda_{db} = 5, \quad \lambda_{EAt} = 250$$
$$\lambda_{EIt} = 6250, \quad \lambda_{EAb} = 250, \quad \lambda_{mt} = 50, \quad \lambda_{mb} = 50$$
$$\lambda_{EIb} = 6250, \quad \lambda_{nt} = 10, \quad \lambda_{nb} = 10, \quad \lambda_{qb} = 10 \tag{31}$$

Equation (31) fully describes the model.

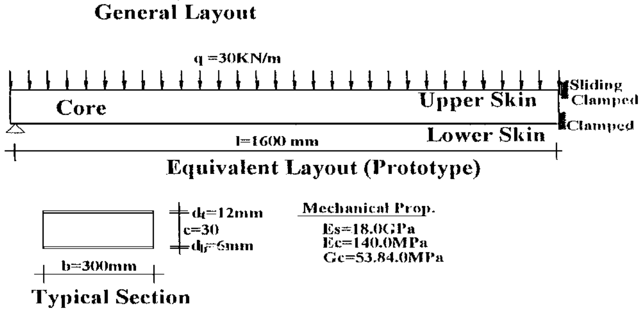


Fig. 2 Geometry, loading, boundary conditions, and mechanical properties of a typical sandwich panel with a soft core.

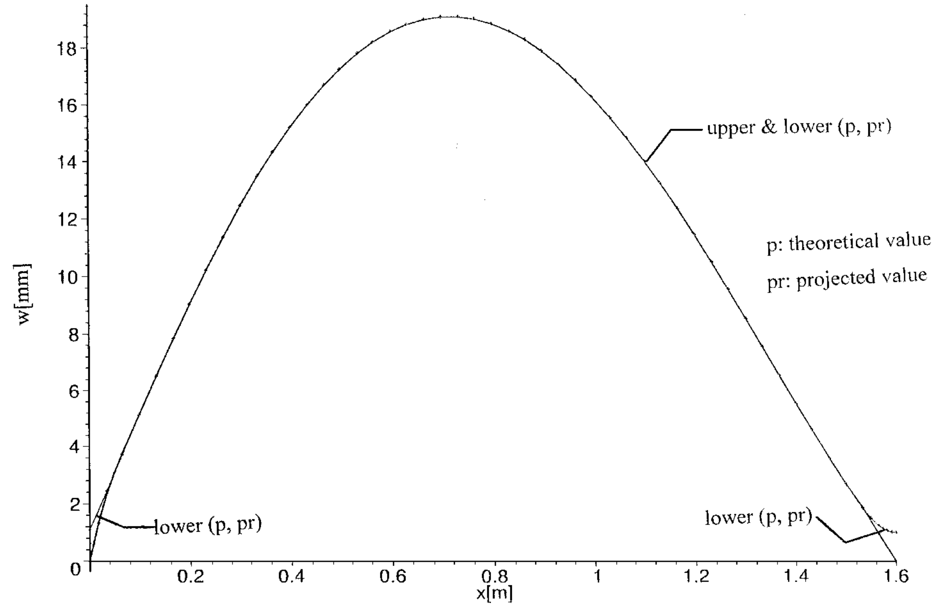


Fig. 3a Vertical and displacements (complete similarity).

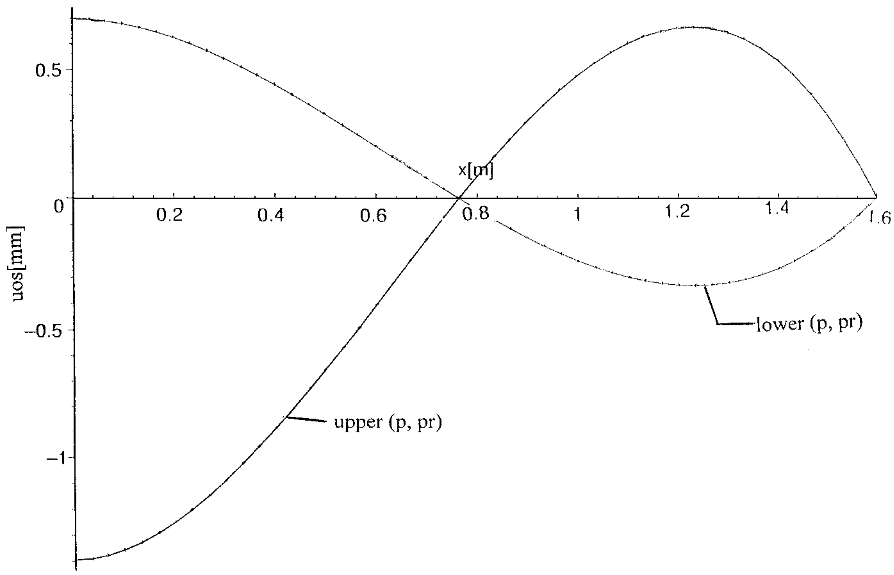


Fig. 3b Longitudinal displacements (complete similarity).

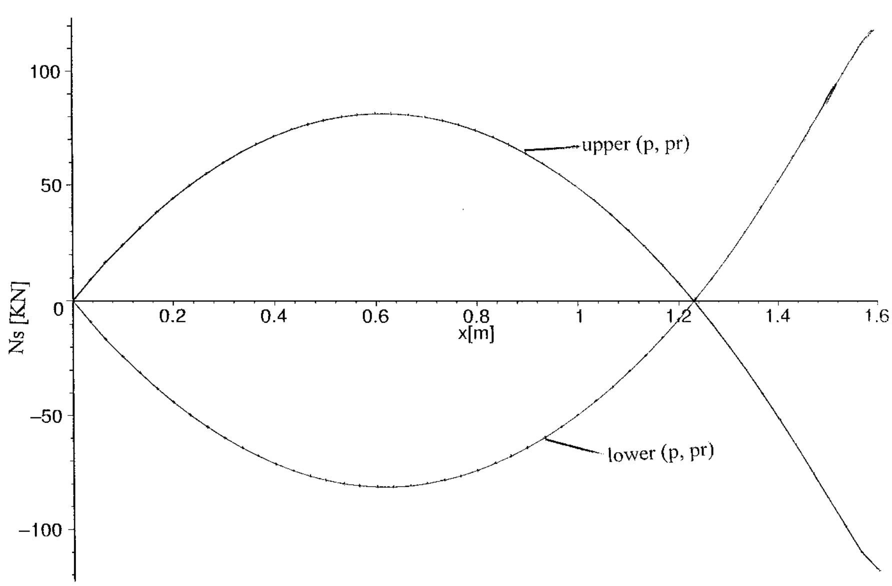


Fig. 3c Longitudinal forces (complete similarity).

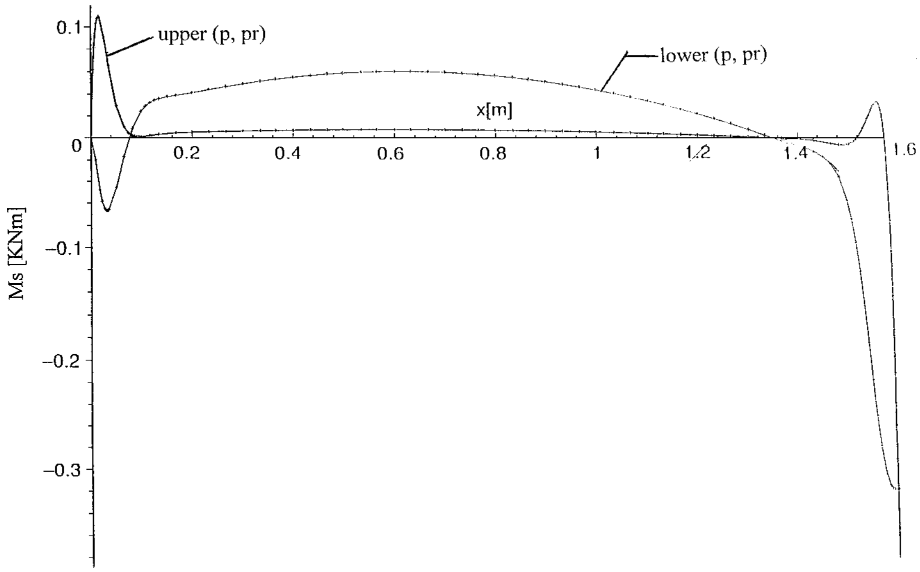


Fig. 3d Bending moments (complete similarity).

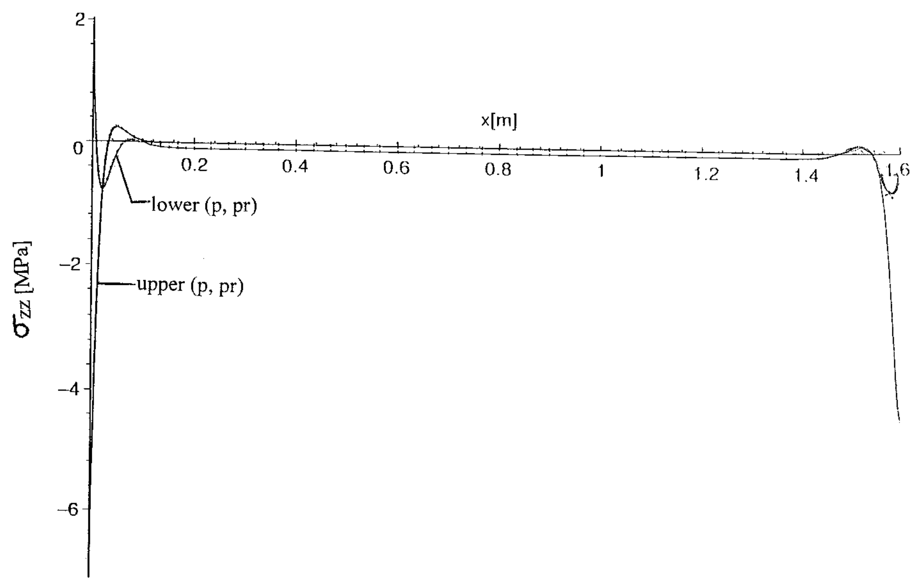


Fig. 4a Vertical normal stresses (complete similarity).

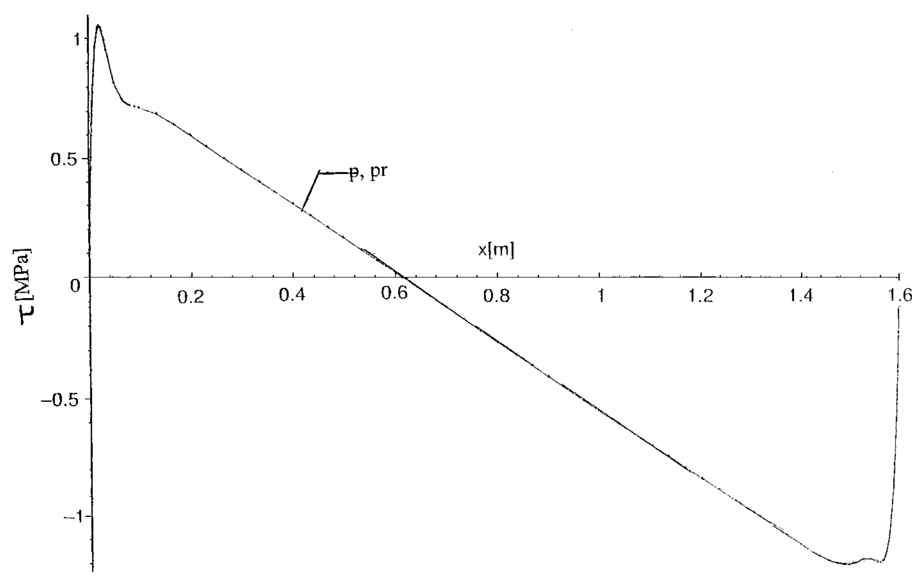


Fig. 4b Shear stresses in core (complete similarity).

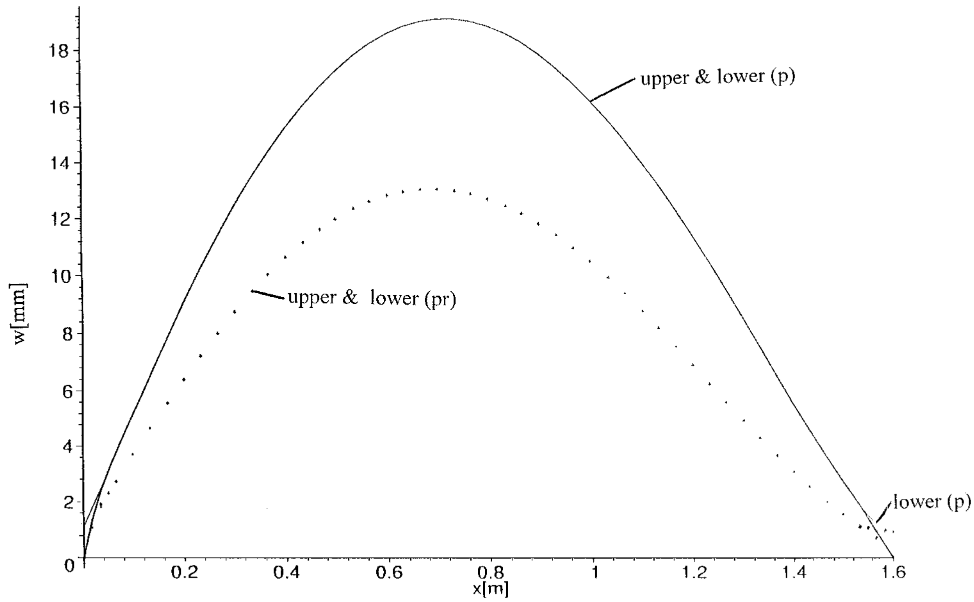


Fig. 5a Vertical displacement (partial similarity).

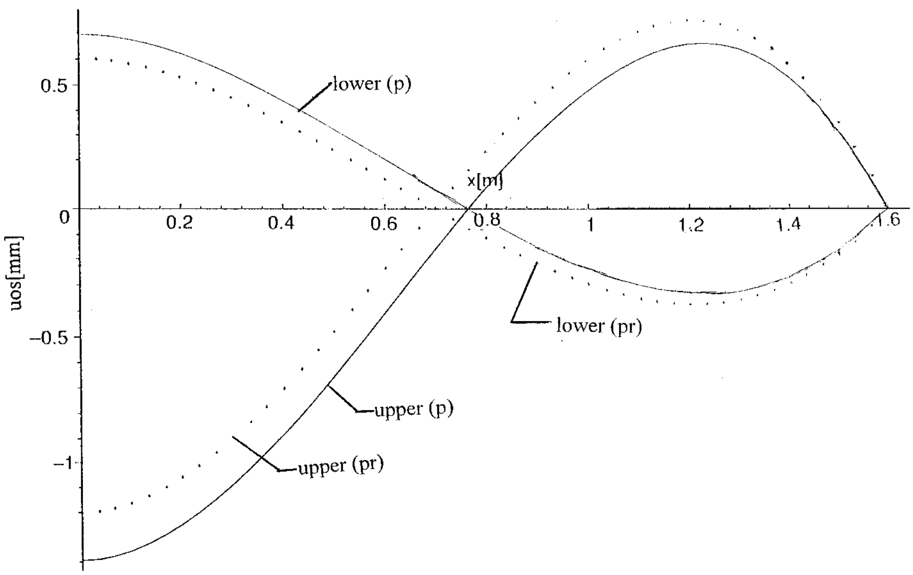


Fig. 5b Longitudinal displacement (partial similarity).

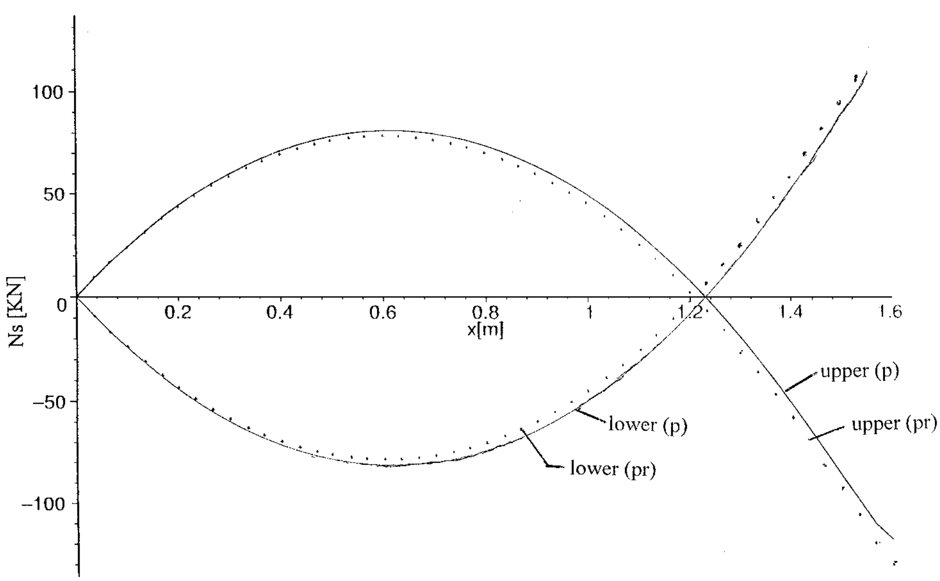


Fig. 5c Longitudinal forces (partial similarity).

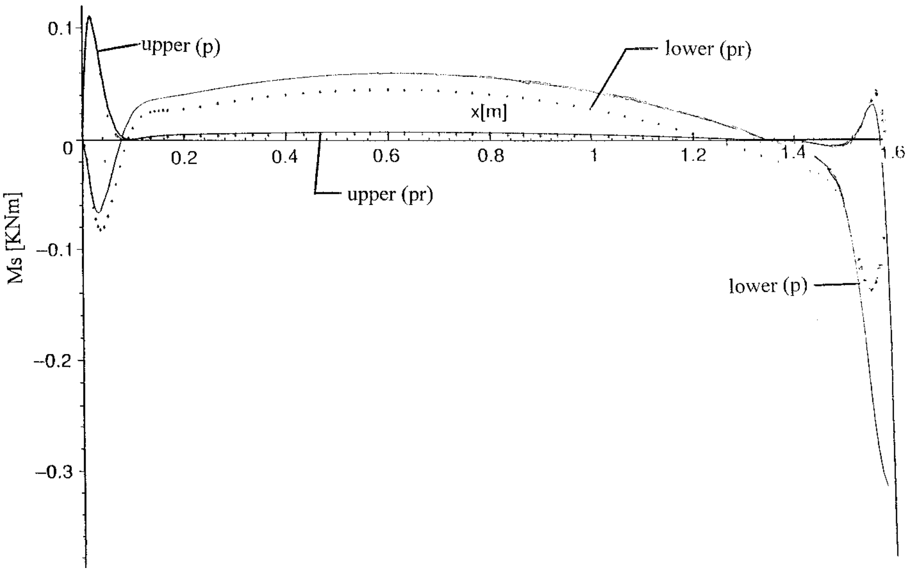


Fig. 5d Bending moments (partial similarity).

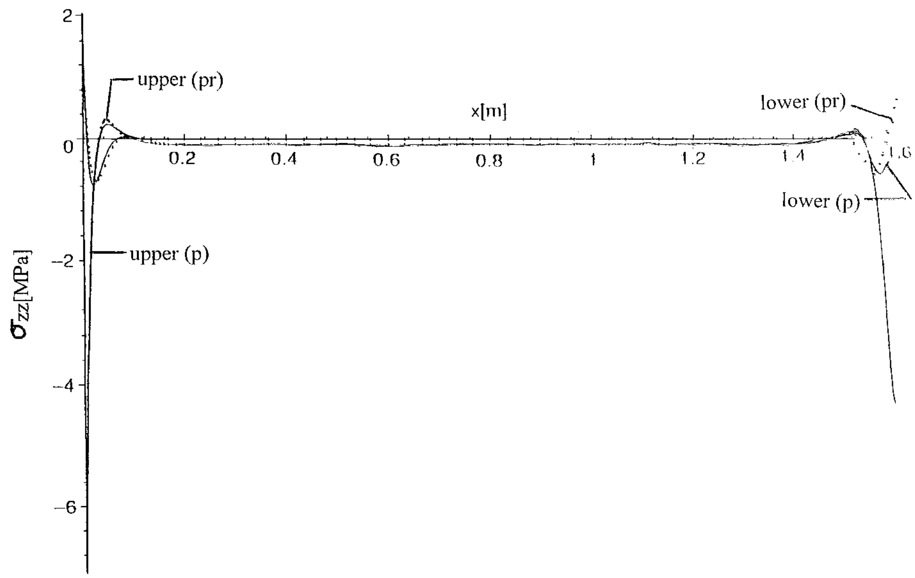


Fig. 6a Vertical normal stresses (partial similarity).

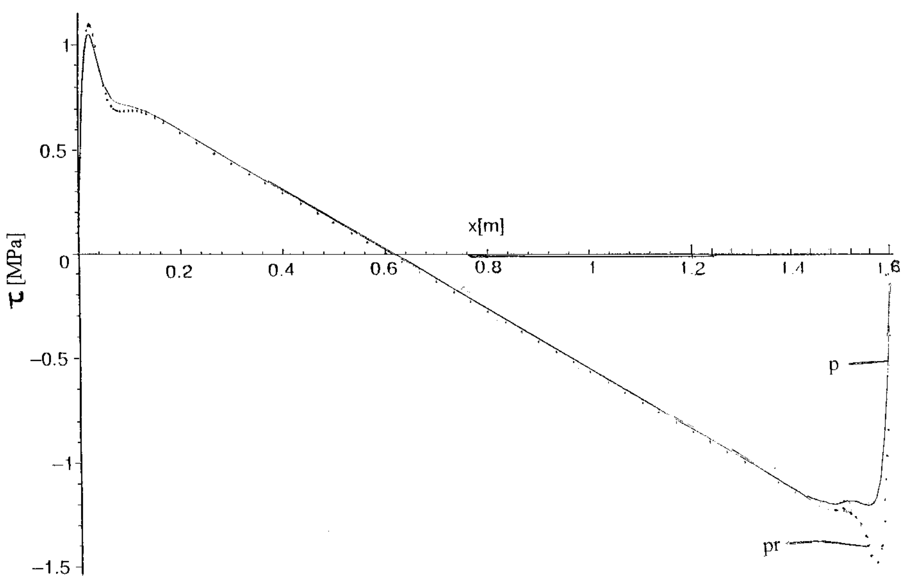


Fig. 6b Shear stresses in core (partial similarity).

In addition, the scale factors for the stress resultants and the core fields are

$$\lambda_{uc} = 1, \quad \lambda_{Nxxt} = 50, \quad \lambda_{Nxxb} = 50, \quad \lambda_{Mxxt} = 250$$
$$\lambda_{Mxxb} = 250, \quad \lambda_{Qxxt} = 50, \quad \lambda_{Qxxb} = 50$$
$$\lambda_{Qxxc} = 50, \quad \lambda_{N\text{tot}} = 50, \quad \lambda_{M\text{tot}} = 250$$
$$\lambda_{Q\text{tot}} = 50, \quad \lambda_{wc} = 1, \quad \lambda_{\sigma_{zzc}} = 2 \tag{32}$$

The results in the form of displacements, stress resultants in the face sheets and in the core, as well as the shear stresses in the core and the vertical normal stresses (peeling) at the face–core interfaces have been determined analytically for the prototype and for the model independently. The model theoretical results are used with the scale factors shown earlier to predict the response of the prototype. Theses predictions are then compared to the analytical results of the prototype. Because all of the similarity conditions are satisfied, we have complete similarity. Both results for the prototype (analytical and predicted) are shown graphically in Figs. 3 and 4. The agreement is excellent.

Complete Similarity (Case 2)

The second model is not a scaled down description of the prototype, and this means $\lambda_x \neq \lambda_c$. Hence, the chosen scale factors are

$$\lambda_c = 1, \quad \lambda_{qt} = 10, \quad \lambda_{Ec} = 1, \quad \lambda_x = 2 \tag{33}$$

The remaining scale factors are

$$\lambda_{uot} = 1, \quad \lambda_{uob} = 1, \quad \lambda_{wt} = 2, \quad \lambda_{wb} = 2, \quad \lambda_\tau = 4$$
$$\lambda_b = 5, \quad \lambda_{dt} = 1, \quad \lambda_{db} = 1, \quad \lambda_{EAt} = 80, \quad \lambda_{EIt} = 80$$
$$\lambda_{EAb} = 80, \quad \lambda_{EIb} = 80, \quad \lambda_{Gc} = 1, \quad \lambda_{nt} = 20$$
$$\lambda_{nb} = 20, \quad \lambda_{qb} = 10, \quad \lambda_{mt} = 20, \quad \lambda_{mb} = 20$$
$$\lambda_{uc} = 1, \quad \lambda_{Nxxt} = 40, \quad \lambda_{Nxxb} = 40, \quad \lambda_{Mxxt} = 40$$
$$\lambda_{Mxxb} = 40, \quad \lambda_{Qxxt} = 20, \quad \lambda_{Qxxb} = 20$$
$$\lambda_{Qxxc} = 20, \quad \lambda_{N\text{tot}} = 40, \quad \lambda_{M\text{tot}} = 40$$
$$\lambda_{Q\text{tot}} = 20, \quad \lambda_{\sigma_{zzc}} = 2, \quad \lambda_{wc} = 2 \tag{34}$$

Also, in this case we have complete similarity. When the same procedure as in case 1 is followed, the predicted and analytical results are shown in Figs. 3 and 4.

Partial Similarity

In this case, the similarity is partial, and the shear modulus ratio of the core does not fulfill the similarity condition. The independent scale factors are those that appear in Eq. (33) and the remaining scale factors are those that appear in Eqs. (34). The shear modulus ratio chosen is equal to

$$\lambda_{Gc} = 4 \tag{35}$$

and not 1, as in the case of complete similarity [see Eqs. (34)].

The results of the prototype (analytical and predicted) appear in Figs. 5 and 6. Here, because there is no complete similarity, the model and the prototype results do not coincide. However, there are some large differences in the vertical deflections but minor differences in the longitudinal deformations, stress resultants and stresses in the core, and at its interfaces.

Conclusions

Structural similitude as applied to sandwich beams has been demonstrated with success, for complete similarity. Partial similarity requires further study.

Note that when using the field equation approach one can easily design a model for a given prototype because the number of similarity conditions is smaller than the number of scaling laws or similarity conditions.

Future work should include application to buckling and vibration problems as well as other geometries.

Acknowledgments

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